EXERCISE 2.1

Q.1 Find the discriminant of the following given quadratic equation.

Solutions:

(i) $2x^2+3x-1=0$

As $ax^2+bx+c=0$

$\Rightarrow$ $a=2$, $b=3$, $c=-1$

$$\text{Disc.}=b^2-4ac$$
$$=(3)^2-4(2)(-1)$$
$$=9+8$$
$$=17$$

(ii) $6x^2-8x+3=0$

As $ax^2+bx+c=0$

$\Rightarrow$ $a=6$, $b=-8$, $c=3$

$$\text{Disc.}=b^2-4ac$$
$$=(-8)^2-4(6)(3)$$
$$=64-72$$
$$=-8$$

(iii) $9x^2-30x+25=0$

As $ax^2+bx+c=0$

$\Rightarrow$ $a=9$, $b=-30$, $c=25$

$$\text{Dis}=b^2-4ac$$
$$=(-30)^2-4(9)(25)$$
$$=900-900$$
$$=0$$

(iv) $4x^2-7x-2=0$

As $ax^2+bx+c=0$

$\Rightarrow$ $a=4$, $b=-7$, $c=-2$

$$\text{Disc}=b^2-4ac$$
$$=(-7)^2-4(4)(-2)$$
$$=49+32$$
$$=81$$

Q.2 Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations

(i) $x^2-23x+120=0$

Solutions:

$$x^2-23x+120=0$$

As $ax^2+bx+c=0$

$\Rightarrow$ $a=1$, $b=-23$, $c=120$

$$\text{Disc}=b^2-4ac$$
$$=(-23)^2-4(1)(120)$$
$$=529-480$$
$$=49$$
$$=(7)^2$$

As Disc. is positive and perfect square, therefore roots of equation are real, rational and un equal.

Verification.

Now solving the Equation

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-(-23)\pm\sqrt{(-23)^2-4(1)(120)}}{2(1)}$$
$$x=\frac{23\pm\sqrt{529-480}}{2}$$
$$x=\frac{23\pm\sqrt{49}}{2}$$
$$x=\frac{23\pm7}{2}$$
$$x=\frac{23+7}{2} \quad \text{or} \quad x=\frac{23-7}{2}$$
$$x=\frac{30}{2} \quad \text{or} \quad x=\frac{16}{2}$$
$$x=15 \quad \text{or} \quad x=8$$

Thus roots are real, rational and unequal.

(ii) $\mathbf{2x^2 + 3x + 7 = 0}$

As $ax^2 + bx + c = 0$

$\Rightarrow$ $a = 2 \quad , b = 3 \quad , c = 7$

$\text{Disc} = b^2 - 4ac$

$= (3)^2 - 4(2)(7)$

$= 9 - 56$

$= -47 < 0$

As Disc. is negative, therefore roots of the equation are imaginary(complex conjugates)

Verification

We verify the results by solving the equation.

$2x^2 + 3x + 7 = 0$

$a = 2 \ , \ b = 3 \ , \ c = 7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(7)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9 - 56}}{4}$$

$$x = \frac{-3 \pm \sqrt{-47}}{4}$$

Thus roots are imaginary.

(iii) $\mathbf{16x^2 - 24x + 9 = 0}$

As $ax^2 + bx + c = 0$

$\Rightarrow$ $a = 16 \quad , \quad b = -24 \quad , \quad c = 9$

$\text{Disc} = b^2 - 4ac$

$= (-24)^2 - 4(16)(9)$

$= 576 - 576$

$= 0$

As Disc. = 0 therefore roots of equation are rational (real) and equal.

Verification:

We verify the result by solving the equation. Here

$a = 16 \ , \ b = -24 \ , \ c = 9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(16)(9)}}{2(16)}$$

$$x = \frac{24 \pm \sqrt{576 - 576}}{32}$$

$$x = \frac{24 \pm \sqrt{0}}{32}$$

$$x = \frac{24 \pm 0}{32}$$

$$x = \frac{24 + 0}{32} \quad \text{or} \quad x = \frac{24 - 0}{32}$$

$$x = \frac{24}{32} \quad \text{or} \quad x = \frac{24}{32}$$

$$x = \frac{3}{4} \quad \text{or} \quad x = \frac{3}{4}$$

Thus roots are rational (real) and equal.

(iv) $\mathbf{3x^2 + 7x - 13 = 0}$

As $ax^2 + bx + c = 0$

$\Rightarrow$ $a = 3 \ , \ b = 7 \ , \ c = -13$

$\text{Disc} = b^2 - 4ac$

$= (7)^2 - 4\ (3)\ (-13)$

$= 49 + 156$

$= 205$

Therefore roots of equation are real, irrational, and unequal.

Verification:

We verify the result by solving the equation.

$\mathbf{3x^2 + 7x - 13 = 0}$

$a=3 \ , \ b=7 \ , \ c = -13$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-13)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 + 156}}{6}$$

$$x = \frac{-7 \pm \sqrt{205}}{6}$$

Thus roots are real, irrational and unequal.

Q.3 For what value of k, the expression $k^2x^2 + 2(k+1)x + 4$ is perfect square.

Solution:

$a = k^2$, $b = 2(k+1)$, $c = 4$

$$b^2 - 4ac = [2(k+1)]^2 - 4(k^2)(4)$$
$$= 2^2(k+1)^2 - 16k^2$$
$$= 4(k^2 + 2k + 1) - 16k^2$$
$$= 4k^2 + 8k + 4 - 16k^2$$
$$= -12k^2 + 8k + 4$$

As given that expression is perfect square therefore

$$Disc = 0$$
$$-12k^2 + 8k + 4 = 0$$
$$-4(3k^2 - 2k - 1) = 0$$
$$\Rightarrow \quad 3k^2 - 2k - 1 = 0 \qquad (\because -4 \neq 0)$$
$$3k^2 - 3k + k - 1 = 0$$
$$3k(k-1) + 1(k-1) = 0$$
$$(k-1)(3k+1) = 0$$

$k - 1 = 0$ or $3k + 1 = 0$

$k = 0 + 1$ or $3k = 0 - 1$

$k = 1$ or $k = \frac{-1}{3}$

So values of k are 1 and $\frac{-1}{3}$

Q.4 Find the value of K, if the roots of the following equations are equal.

Solution:

(i) $(2k-1)x^2 + 3kx + 3 = 0$

As $ax^2 + bx + c = 0$

$a = 2k - 1$, $b = 3k$, $c = 3$

$$\text{Disc.} = b^2 - 4ac$$
$$= (3k)^2 - 4(2k-1)(3)$$
$$= 9k^2 - 12(2k-1)$$
$$= 9k^2 - 24k + 12$$

As the roots of given equation are equal, So

$$\text{Disc.} = 0$$
$$9k^2 - 24k + 12 = 0$$
$$3(3k^2 - 8k + 4) = 0$$
$$3k^2 - 8k + 4 = 0 \qquad (\because 3 \neq 0)$$
$$3k^2 - 6k - 2k + 4 = 0$$
$$3k(k-2) - 2(k-2) = 0$$
$$(k-2)(3k-2) = 0$$

Either $k - 2 = 0$ or $3k - 2 = 0$

$k = 2$ or $3k = 2$

or $k = \frac{2}{3}$

So values of k are 2 and $\frac{2}{3}$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

Solution:

As $ax^2 + bx + c = 0$

$a = 1$, $b = 2(k+2)$, $c = 3k + 4$

$$\text{Disc.} = b^2 - 4ac$$
$$= [2(k+2)]^2 - 4(1)(3k+4)$$
$$= 4(k+2)^2 - 4(3k+4)$$
$$= 4(k^2 + 4k + 4) - 12k - 16$$
$$= 4k^2 + 16k + \cancel{16} - 12k - \cancel{16}$$
$$= 4k^2 + 4k$$
$$= 4k(k+1)$$

As roots of given equation are equal, so

$$\text{Disc.} = 0$$
$$4k(k+1) = 0$$

Either $4k = 0$ or $k + 1 = 0$

As $4 \neq 0$ or $k = -1$

$\Rightarrow k = 0$

So values of k are 0 and -1

(iii) $(3k+2)x^2 - 5(k+1)x + (2k+3) = 0$

Solution:

As $ax^2 + bx + c = 0$

$a = (3k+2)$ $b = -5(k+1)$, $c = 2k + 3$

$$\text{Disc} = b^2 - 4ac$$
$$= [-5(k+1)]^2 - 4(3k+2)(2k+3)$$
$$= (-5)^2(k+1)^2 - 4(6k^2 + 9k + 4k + 6)$$
$$= 25[k^2 + 2(k)(1) + 1^2] - 4(6k^2 + 13k + 6)$$

$= 25k^2 + 50k + 25 - 24k^2 - 52k - 24$

$= k^2 - 2k + 1$

$= k^2 - k - k + 1$

$= k(k-1) - 1(k-1)$

$= (k-1)(k-1)$

As root of given equation are equal

Disc = 0

$(k-1)(k-1) = 0$

$k - 1 = 0$ or $k - 1 = 0$

$k = 1$ or $k = 1$

So values of k is 1

Q.5 Show that the equation $x^2 + (mx+c)^2 = a^2$ has equal roots if $c^2 = a^2(1+m^2)$

Solution: We have given

$x^2 + (mx+c)^2 = a^2$

$x^2 + (mx)^2 + (c)^2 + 2(mx)(c) = a^2$

$x^2 + m^2x^2 + c^2 + 2mcx - a^2 = 0$

$(1+m^2)x^2 + 2mc\,x + c^2 - a^2 = 0$

$Ax^2 + Bx + C = 0$

$A = (1+m^2),\ B = 2mc,\ C = c^2 - a^2$

Disc $= B^2 - 4AC$

$= (2mc)^2 - 4(1+m^2)(c^2 - a^2)$

$= 4m^2c^2 - 4(c^2 - a^2 + m^2c^2 - a^2m^2)$

$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4a^2m^2$

$= -4c^2 + 4a^2 + 4a^2m^2$

As roots of the equation are equal,

So Disc = 0

$-4c^2 + 4a^2 + 4a^2m^2 = 0$

$-4\left[c^2 - a^2 - a^2m^2\right] = 0$

As $-4 \neq 0$, so $c^2 - a^2 - a^2m^2 = 0$

$c^2 = a^2 + a^2m^2$

$c^2 = a^2(1+m^2)$

Hence proved

Q.6 Find the condition that the roots of the equation $(mx+c)^2 - 4ax = 0$ are equal

Solution: We have given

$(mx+c)^2 - 4ax = 0$

$(mx^2) + (c)^2 + 2(mx)(c) - 4ax = 0$

$m^2x^2 + c^2 + 2mcx - 4ax = 0$

$m^2x^2 + (2mc - 4a)x + c^2 = 0$

$Ax^2 + Bx + C = 0$

$A = m^2,\ B = 2mc - 4a,\ C = c^2$

Disc. $= B^2 - 4AC$

$= (2mc - 4a)^2 - 4(m^2)(c^2)$

$= (2mc)^2 + (4a)^2 - 2(2mc)(4a) - 4m^2c^2$

$= 4m^2c^2 + 16a^2 - 16acm - 4m^2c^2$

$= 16a^2 - 16acm$

$= 16a(a - mc)$

As root of the equation are equal,

So Disc = 0

$16a(a - mc) = 0$

Either $16a = 0$ or $a - mc = 0$

As $16 \neq 0$ So $a = 0$ or $a = mc$

Thus required condition is a=0 or a=mc

Q.7 If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ or $a^3 + b^3 + c^3 = 3abc$

Solution: We have given

$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$

$Ax^2 + Bx + C = 0$

$A = (c^2 - ab),\ B = -2(a^2 - bc),\ C = (b^2 - ac)$

Disc. $= B^2 - 4AC$

$= \left[-2(a^2 - bc)\right]^2 - 4(c^2 - ab)(b^2 - ac)$

$= (-2)^2(a^2 - bc)^2 - 4(c^2 - ab)(b^2 - ac)$

$= 4\left[(a^2)^2 + (bc)^2 - 2(a^2)(bc)\right] - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$

$= 4(a^4 + b^2c^2 - 2a^2bc) - 4(b^2c^2 - ac^3 - ab^3 + a^2bc)$

$= 4a^4 + 4b^2c^2 - 8a^2bc - 4b^2c^2 + 4ac^3 + 4ab^3 - 4a^2bc$

$$= 4a^4 + 4ac^3 + 4ab^3 - 12a^2bc$$
$$= 4a(a^3 + b^3 + c^3 - 3abc)$$

If roots of given equation are equal, then

Disc. = 0

$$4a(a^3 + b^3 + c^3 - 3abc) = 0$$

$$4a = 0 \quad \text{or} \quad a^3 + b^3 + c^3 - 3abc = 0$$

As $4 \neq 0$, or $\boxed{a^3 + b^3 + c^3 = 3abc}$

So, $a = 0$

Q.8 Show that the roots of the following equations are rational

Solution:

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

$$Ax^2 + Bx + C = 0$$
$$A = a(b-c),\ B = b(c-a),\ C = c(a-b)$$

Disc.$= B^2 - 4AC$

$$= [b(c-a)]^2 - 4[a(b-c)c(a-b)]$$
$$= b^2(c-a)^2 - 4[ac(b-c)(a-b)]$$
$$= b^2(c^2 + a^2 - 2ac) - 4ac(ab - b^2 - ca + bc)$$
$$= b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc + 4ab^2c + 4c^2a^2 - 4abc^2$$
$$= a^2b^2 + b^2c^2 + 4c^2a^2 + 2ab^2c - 4abc^2 - 4a^2bc$$
$$= (ab)^2 + (bc)^2 + (-2ca)^2 + 2(ab)(bc) + 2(bc)(-2ca) + 2(-2ca)(ab)$$

By using formula

$$(A+B+C)^2 = A^2 + B^2 + C^2 + 2AB + 2BC + 2CA$$

So, $= [(ab) + (bc) + (-2ca)]^2$

$$= (ab + bc - 2ca)^2 > 0$$

As Disc. is perfect square so the roots of equations are rational.

(ii) $(a+2b)\ x^2 + 2(a+b+c)\ x + (a+2c) = 0$

Solution:

A = a+2b , B=2 (a+b+c) , C= (a + 2c)

Disc. $= B^2 - 4AC$

$$= [2(a+b+c)]^2 - 4\,(a+2b)\,(a+2c)$$
$$= 4[a^2+b^2+c^2+2ab+2bc+2ca] - 4(a^2+2ac+2ab+4bc)$$
$$= 4a^2 + 4b^2 + 4c^2 + 8ab + 8bc + 8ca - 4a^2 - 8ac - 8ab - 16bc$$
$$= 4b^2 + 4c^2 + 8bc - 16bc$$
$$= 4b^2 + 4c^2 - 8bc$$
$$= (2b)^2 + (2c)^2 - 2\,(2b)\,(2c)$$
$$= (2b - 2c)^2 > 0$$

As Disc. is perfect square therefore roots, are rational (real and unequal)

Q.9. For all values of K, Prove that the Roots, of the Equation.

$$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0,\ k \neq 0 \text{ are real}$$

Solution. $a=1$, $b - 2\left(k + \frac{1}{k}\right)$, $c = 3$

Disc. $= b^2 - 4ac$

$$= \left[-2\left(k + \frac{1}{k}\right)\right]^2 - 4(1)(3)$$
$$= 4\left(k + \frac{1}{k}\right)^2 - 12$$
$$= 4\left[\left(k + \frac{1}{k}\right)^2 - 3\right]$$
$$= 4\left[k^2 + \frac{1}{k^2} + 2(k)\left(\frac{1}{k}\right) - 3\right]$$
$$= 4\left[k^2 + \frac{1}{k^2} + 2 - 3\right]$$
$$= 4\left[k^2 + \frac{1}{k^2} - 1\right]$$
$$= 4\left[(k)^2 + \left(\frac{1}{k}\right)^2 - 2 + 1\right]$$
$$= 4\left[(k)^2 + \left(\frac{1}{k}\right)^2 - 2(k)\left(\frac{1}{k}\right) + 1\right]$$
$$= 4\left[\left(k - \frac{1}{k}\right)^2 + 1\right] > 0$$

As Disc. is positive so roots of the given equation are real.

Q.10. Show that the roots of the equation $(b - c)x^2+(c - a)x+(a - b) = 0$ are real.

Solution: $(b - c)x^2+(c - a)x+(a - b) = 0$

$Ax^2 + Bx + C= 0$

$A = (b - c)$, $B = (c - a)$, $C = (a - b)$

$\text{Disc}= B^2 - 4AC$

$= (c - a)^2-4(b - c)(a - b)$

$= (c^2+a^2-2ca) -4 (ab-b^2-ca+bc)$

$= c^2+a^2-2ca-4ab+4b^2+4ca-4bc$

$= c^2+a^2+4b^2+2ca-4bc-4ab$

$= a^2 + 4b^2 + c^2 - 4ab - 4bc + 2ca$

$= (a)^2+(-2b)^2+(c)^2+2(a)(-2b)+2(-2b)(c)+2(c)(a)$

$= (a - 2b + c)^2$

Perfect square shows that the roots of the given equation are real.